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LETTER TO THE EDITOR

Symmetries for the super-KdV equation

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Abstract. Symmetries and higher-order or generalised symmetries for the SKdV equation are constructed. Moreover by the introduction of graded potentials a non-local generalised symmetry is obtained, leading to the recursion operator for symmetries in a straightforward way, without making use of the bi-Hamiltonian structure.

In the study of complete integrability of classical evolution equations like KdV, MKdV, Boussinesque, massive Thirring and other well known equations there was a great emphasis on Wahlquist–Estabrook prolongation and higher-order or generalised symmetry calculations [1–3].

We constructed computer algebra programs to handle the enormous computations arising from these concepts. As a step towards supersymmetric equations we recently constructed a graded differential geometry package in REDUCE [4].

The present letter treats the first interesting problem handled completely by the developed software.

The notions of graded differential geometry are taken from Kostant [5] while the graded jet bundle formulation is due to Hernández Ruiperez and Muñoz Masqué [6].

First, the SKdV equation [7] is cast into a graded exterior differential system I. Graded infinitesimal symmetries which are vector fields that leave invariant the exterior differential system, i.e.

$$L_V I \subset I$$

are obtained and the result is given.

Next, the first higher-order symmetry is constructed, satisfying a similar condition to that in the classical case [8], i.e.

$$L_V(\mathrm{D}^\infty I) \subset \mathrm{D}^\infty I$$

where $\mathrm{D}^\infty I$ is the infinite prolongation of the graded ideal I. Following [9], non-local variables are introduced in the graded case and a non-local x, t-dependent higher-order symmetry of SKdV equation is then obtained.

Finally we derive the Lenard recursion operator for higher-order symmetries of the SKdV equation.

The SKdV equation is given by the following system of graded partial differential equations [7]:

$$\begin{aligned} u_t &= 6uu_x - u_{xxx} + 3\phi\phi_{xx} \\ \phi_t &= 3u_x\phi + 6u\phi_x - 4\phi_{xxx} \end{aligned} \qquad (1)$$

where subscripts denote partial derivatives; t is the time and x is the space variable; u, x and t are even (commuting), while ϕ is odd (anticommuting). We shall often use graded instead of super, following Kostant.

We now construct the graded exterior differential system I associated with (1), generated by the 2-forms

$$\begin{aligned}
\alpha_1 &= -\mathrm{d}t\,\mathrm{d}u + \mathrm{d}x\,\mathrm{d}t(-u_x)\\
\alpha_2 &= -\mathrm{d}t\,\mathrm{d}u_x + \mathrm{d}x\,\mathrm{d}t(-u_{xx})\\
\alpha_3 &= -\mathrm{d}x\,\mathrm{d}u - \mathrm{d}t\,\mathrm{d}u(6u) + \mathrm{d}t\,\mathrm{d}u_{xx} + \mathrm{d}t\,\mathrm{d}\phi_x(3\phi)\\
\alpha_4 &= -\mathrm{d}t\,\mathrm{d}\phi + \mathrm{d}x\,\mathrm{d}t(-\phi_x)\\
\alpha_5 &= -\mathrm{d}t\,\mathrm{d}\phi_x + \mathrm{d}x\,\mathrm{d}t(-\phi_{xx})\\
\alpha_6 &= -\mathrm{d}x\,\mathrm{d}\phi - \mathrm{d}t\,\mathrm{d}u(3\phi) - \mathrm{d}t\,\mathrm{d}\phi(6u) + \mathrm{d}t\,\mathrm{d}\phi_{xx}(4).
\end{aligned} \tag{2}$$

defined on $\boldsymbol{R}^{(5,3)} = \{(x, t, u, u_x, u_{xx}; \phi, \phi_x, \phi_{xx})\}$. Throughout this letter we shall use the right module structure of differential forms.

We now search for vector fields V defined on $\boldsymbol{R}^{(5,3)}$ which satisfy the infinitesimal symmetry condition

$$L_V I \subset I \tag{3}$$

where L_V denotes the (graded) Lie derivative with respect to the vector field V.

It is easy to see that the vector fields V satisfying (3) constitute a graded vector space; in effect a graded Lie algebra. So we can restrict ourselves in the search for solutions of (3) to even and odd vector fields.

Condition (3) leads to overdetermined systems of partial differential equations for the coefficients of the vector field V. Solving these over-determined systems we obtain the following result.

Theorem 1. The graded Lie algebra of infinitesimal symmetries of the sKdV equation (2.1) is four-dimensional and generated by the vector fields

$$\begin{aligned}
&V_1 = \partial_x \qquad V_2 = \partial_t \qquad V_3 = t\partial_x - \tfrac{1}{6}\partial_u\\
&V_4 = -x\partial_x - 3t\partial_t + 2u\partial_u + \tfrac{3}{2}\phi\partial_\phi.
\end{aligned} \tag{4}$$

The vector fields $V_1, \ldots, V_4$ are even; there are no odd infinitesimal symmetries, so the graded Lie algebra of infinitesimal symmetries is just an ordinary Lie algebra.

Classical higher-order symmetries are defined on the infinite jet bundle $J^\infty(x, t, u, \phi)$ [8] and satisfy the symmetry condition

$$L_V \mathrm{D}^\infty I \subset \mathrm{D}^\infty I \tag{5}$$

where $\mathrm{D}^\infty I$ is the infinite prolongation of the exterior differential system I by means of the action of the total partial derivative vector fields D_x, D_t defined by

$$\begin{aligned}
\mathrm{D}_x &= \partial_x + u_x\partial_u + \phi_x\partial_\phi + u_{xx}\partial_{u_x} + \ldots\\
\mathrm{D}_t &= \partial_t + u_t\partial_u + \phi_t\partial_\phi + u_{xt}\partial_{u_x} + \ldots.
\end{aligned} \tag{6}$$

Due to the fact that equations (6) satisfy (5) in an obvious way, the search for higher-order or generalised symmetries can be restricted to vertical vector fields; i.e. the components of ∂_x, ∂_t are taken to be zero.

The vertical vector fields are proved to have the following representation:

$$V = f\partial_u + g\partial_\phi + (\mathrm{D}_x f)\partial_{u_x} + (\mathrm{D}_x g)\partial_{\phi_x} + \ldots \tag{7}$$

so we are only interested in the defining functions f, g of the vector field. The functions f, g are assumed to depend on a finite number of independent variables of the infinite jet bundle.

In the present graded case we proceed in a similar way, keeping in mind the left module structure of the vector fields.

We restrict our search for higher-order symmetries to *even* vector fields; moreover our search is for vector fields V whose defining functions f, g (7) depend on $x, t, u, \phi, \ldots, u_{xxxxx}, \phi_{xxxxx}$, the other components being obtained by prolongation (7). The vector field V has to satisfy the symmetry condition (5) which is equivalent to

$$\begin{aligned} &L_V(u_t - 6uu_x + u_{xxx} - 3\phi\phi_{xx}) \equiv 0 \\ &L_V(\phi_t - 3u_x\phi - 6u\phi_x + 4\phi_{xxx}) \equiv 0 \end{aligned} \tag{8}$$

whereas '$\equiv 0$' should be read as equal to zero on the submanifold in the infinite jet bundle $J^\infty(x, t; u, \phi)$ defined by (1) and its differential consequences. Condition (8) leads to an overdetermined system of partial differential equations for the function f, g including the exterior algebra defined on $\phi, \phi_x, \phi_{xx}, \phi_{xxx}, \phi_{xxxx}, \phi_{xxxxx}$.

From now on we shall write

$$u_i = \underbrace{u_{x\ldots x}}_{i\text{ times}} \qquad \phi_j = \underbrace{\phi_{x\ldots x}}_{j\text{ times}} . \tag{9}$$

Using the developed integration package we obtain the following result. There are five even vector fields satisfying (8) under the above-mentioned assumptions, i.e.

$$\begin{aligned} \tilde{V}_1 &= u_1\partial_u + \phi_1\partial_\phi + \ldots \\ \tilde{V}_2 &= (6uu_1 - u_3 + 3\phi\phi_2)\partial_u + (3u_1\phi + 6u\phi_1 - 4\phi_3)\partial_\phi + \ldots \\ \tilde{V}_3 &= (6tu_1 + 1) + 6t\phi_1\partial_\phi + \ldots \\ \tilde{V}_4 &= \{3t(6uu_1 - u_3 + 3\phi\phi_2) + xu_1 + 2\mu\}\partial_u + \{3t(3u_1\phi + 6u\phi_1 - 4\phi_3) + x\phi_1 + \tfrac{3}{2}\phi\}\partial_\phi + \ldots \\ \tilde{V}_5 &= (u_5 - 10u_3u - 20u_2u_1 + 30u_1u^2 - 15\phi\phi_4 - 10\phi_1\phi_3 + 30u_1\phi\phi_1 + 30u\phi\phi_2)\partial_u \\ &\quad + (16\phi_5 - 40u\phi_3 - 60u_1\phi_2 - 50u_2\phi_1 + 30u^2\phi_1 + 30u_1u\phi - 15u_3\phi)\partial_\phi + \ldots . \end{aligned} \tag{10}$$

Note that the vector fields $\tilde{V}_1, \ldots, \tilde{V}_4$ are equivalent [3] to $V_1, \ldots, V_4$ obtained earlier.

In order to construct the recursion operator for higher-order symmetries we introduce non-local variables. They can be introduced by prolongation of the exterior differential system I or $D^\infty I$ by means of potential forms or equivalently by prolongation of the total partial derivatives $\mathrm{D}_x, \mathrm{D}_t$. For details the reader is referred to [3, 9].

We first construct the potential forms associated with the vector fields $\tilde{V}_1$, $\tilde{V}_2$, i.e. p_1, p_2 defined by

$$\begin{aligned} &\mathrm{d}p_1 - \mathrm{d}x(u) - \mathrm{d}t(3u^2 - u_2 + 3\phi\phi_1) \\ &\mathrm{d}p_2 - \mathrm{d}x(u^2 + 3\phi\phi_1) - \mathrm{d}t(4u^3 + u_1^2 - 2uu_2 + 12u\phi\phi_1 + 8\phi_1\phi_2 - 4\phi\phi_3) \end{aligned} \tag{11}$$

or, equivalently,

$$p_1 = \int_{-\infty}^{x} u\,\mathrm{d}x \qquad p_2 = \int_{-\infty}^{x} (u^2 + 3\phi\phi_1)\,\mathrm{d}x \tag{12}$$

whereas in (12) p_1 and p_2 are to be considered in a formal way. Motivated by the results obtained for the classical (ungraded) KdV equation our search is for a non-local vector field V of the following form:

$$V = C_1 t\tilde{V}_4\alpha_1 + C_2 x\tilde{V}_2 + C_3 p_1\tilde{V}_1 + p_2 V^* + V^{**} \tag{13}$$

where $\tilde{V}_4$, $\tilde{V}_2$ and $\tilde{V}_1$ are defined by (10), C_1, C_2 and C_3 are constants and V^*, V^{**} are vector fields, the defining functions of which have to be determined. The prolongation of the vector field V (13) towards the variables $\partial_{u_1}, \partial_{\phi_1}, \ldots$ is determined by the action of the prolonged total partial derivative vector fields $\tilde{\mathrm{D}}_x$, $\tilde{\mathrm{D}}_t$ where

$$\begin{aligned}\tilde{\mathrm{D}}_x &= \mathrm{D}_x + \mu\partial_{p_1} + (u^2 + 3\phi\phi_1)\partial_{p_2}\\ \tilde{\mathrm{D}}_t &= \mathrm{D}_t + (3u^2 - u_2 + 3\phi\phi_1)\partial_{p_1} + (4u^3 + u_1^2 - 2uu_2 + 12u\phi\phi_1 + 8\phi_1\phi_2 - 4\phi\phi_3)\partial_{p_2}.\end{aligned} \tag{14}$$

We now apply the symmetry condition including the non-local variables p_1, p_2, i.e.

$$\begin{aligned}&L_V(u_t - 6uu_1 + u_3 - 3\phi\phi_2) \equiv 0\\ &L_V(\phi_t - 3u_1\phi - 6u\phi_1 + 4\phi_3) \equiv 0\end{aligned} \tag{15}$$

where now by '$\equiv 0$' we mean vanishing of the Lie derivative on the submanifold of $J(x, t; u, \phi \,|\, p_1, p_2) = \{(x, t, u, \phi, p_1, p_2, u_1\phi_1, \ldots\}$ defined by (1) and its differential consequences, together with

$$\begin{aligned}p_{1x} &= u & p_{1t} &= 3u^2 - u_2 + 3\phi\phi_1\\ p_{2x} &= u^2 + 3\phi\phi_1 & p_{2t} &= 4u^3 + u_1^2 - 2uu_2 + 12u\phi\phi_1 + 8\phi_1\phi_2 - 4\phi\phi_3.\end{aligned} \tag{16}$$

Conditions (15) lead to an overdetermined system of partial differential equations for the defining functions of V^* and V^{**} whose dependency on the jet variables $u_1, \phi_1, \ldots$ is induced by the standard grading of sKdV equations, i.e.

$$\begin{aligned}&\deg(x) = -1 \qquad \deg(t) = 3 \qquad \deg(u) = 2 \qquad \deg(\phi) = \tfrac{3}{2}\\ &\deg(p_1) = 1 \qquad \deg(p_2) = 3.\end{aligned}$$

We are searching for a vector field V, whose $\partial_u, \partial_\phi$ components are of degree 4 and $3\frac{1}{2}$ respectively. Solving the overdetermined system of partial differential equations leads to the following result.

Theorem 2. The vector field V defined by

$$V = -\tfrac{3}{4}t\tilde{V}_5 - \tfrac{1}{4}x\tilde{V}_2 - \tfrac{1}{2}p_1\tilde{V}_1 + V^{**} \tag{17a}$$

where V^{**} is given by

$$V^{**} = (u_2 - 2u^2 - \tfrac{3}{2}\phi\phi_1)\partial_u + (\tfrac{7}{2}\phi_2 - 3u\phi)\partial_\phi \tag{17b}$$

is a non-local higher-order symmetry (even) of the sKdV equation.

Note. From sKdV equation (1) and (16) including differential consequences the coefficients of x, t, p_1, p_2 in (13) can be proved to be symmetries.

In order to compute the Lie bracket of the vector fields $\tilde{V}_1, \ldots, \tilde{V}_4$, V we have to prolong the results (10) towards the non-local variables. This leads to results similar to the results obtained, in the ungraded case, by one of the authors for the massive Thirring model. We shall not continue in this direction but shall construct next the recursion operator for higher-order symmetries leading to the commuting flows.

In the case of the classical KdV equation, i.e. the ungraded case ($\phi=0$), the Lenard recursion operator is obtained by a construction based on a non-local vector field, i.e. the ungraded analogue of the vector field V and the Hamiltonian structure of the KdV equation [10, 11].

The construction of the recursion operator for symmetries of the sKdV equation can be obtained in a similar way and is given below.

A formal proof of its properties and the fact that the higher-order symmetries commute is beyond the scope of this letter and will be published elsewhere.

The sKdV equation (1) can be written in the following Hamiltonian form:

$$\begin{pmatrix} u_t \\ \phi_t \end{pmatrix} = \begin{pmatrix} \partial_x & 0 \\ 0 & 1 \end{pmatrix} \begin{matrix} \delta/\delta_u \\ \\ \delta/\delta_\phi \end{matrix} (u^3 + \tfrac{1}{2}u_x^2 + 3u\phi\phi_x - 2\phi\phi_{xxx}) \tag{18}$$

where all variational derivatives are taken to be left ones. In (5.1)

$$\Omega^{-1} = \begin{pmatrix} \partial_x & 0 \\ 0 & 1 \end{pmatrix} \qquad \Omega = \begin{pmatrix} \mathrm{D}^{-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathrm{D}^{-1} = \int_{-\infty}^{x} \cdot\, \mathrm{d}x \tag{19}$$

are analogous to the simplectic operator and its inverse. We now proceed in a way similar to the ungraded case and calculate the variational derivative of ΩV, i.e. $(\Omega V)'$, and its adjoint $(\Omega V)'^*$. Then we are led to the recursion operator for symmetries by (cf [10, 11])

$$T = \Omega^{-1}\{(\Omega V)' - (\Omega V)'^*\}. \tag{20}$$

A simple and straightforward computation starting from V (17a, b) and (19) results in the following:

$$\begin{aligned} T &= \begin{pmatrix} \mathrm{D} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{D} - 2u\mathrm{D}^{-1} - 2\mathrm{D}^{-1}(u) & 2\mathrm{D}^{-1}(\phi_1) - 3\phi \\ -2\phi_1\mathrm{D}^{-1} - 3\phi & 4\mathrm{D}^2 - 4u \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{D}^2 - 2u_1\mathrm{D}^{-1} - 4u & -\phi_1 - 3\phi\mathrm{D} \\ -2\phi_1\mathrm{D}^{-1} - 3\phi_1 & 4\mathrm{D}^2 - 4u \end{pmatrix}. \end{aligned} \tag{21}$$

An easy computation verifies

$$T\begin{pmatrix} u_1 \\ \phi_1 \end{pmatrix} = \begin{pmatrix} u_3 - 6uu_1 - 3\phi\phi_2 \\ 4\phi_3 - 6u\phi_1 - 3u_1\phi \end{pmatrix} \tag{22a}$$

i.e.

$$T(\tilde{V}_1) = -\tilde{V}_2 \qquad T(\tilde{V}_2) = -\tilde{V}_5. \tag{22b}$$

The use of computer programs to handle graded differential geometry computations, and the notion of non-local graded symmetries, leads in a constructive way to the recursion operator for symmetries of the sKdV equation.

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